

**AHARONOV-BOHM EFFECT
IN THE ABELIAN HIGGS THEORY***

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ABSTRACT

We study a field-theoretical analogue of the Aharonov-Bohm effect in the Abelian Higgs Model: the corresponding topological interaction is proportional to the linking number of the Abrikosov-Nielsen-Olesen string world sheets and the particle world trajectory. The creation operators of the strings are explicitly constructed in the path integral and in the Hamiltonian formulation of the theory. We show that the Aharonov-Bohm effect gives rise to several nontrivial commutation relations. We also study the Aharonov-Bohm effect in the lattice formulation of the Abelian Higgs Model. It occurs that this effect gives rise to a nontrivial interaction of tested charged particles.

1. Introduction

In this talk we show that the Aharonov-Bohm effect [1] is responsible for nontrivial effects in quantum Abelian Higgs theory.

It is well known that the four-dimensional Abelian Higgs theory obeys the topologically stable classical solutions called Abrikosov-Nielsen-Olesen (ANO) [2] strings. These strings carry the quantized magnetic flux, wave function of charged particle which scatters the ANO string acquires additional phase. This shift of the phase can be measured and this is the physical effect, which is the analogue of the Aharonov-Bohm effect: the ANO strings playing the role of solenoids which scatter the charged particles. The topological long-range interaction of the Aharonov-Bohm type between the strings and charged particles was discussed in Refs. [3, 4, 5, 6], the same effect in the string representation of the 4D Abelian Higgs model was considered on the lattice and in the continuum limit in Refs. [7, 8]. The problem of ANO string quantization was considered in Refs. [8, 9].

The Aharonov-Bohm effect in the three- and two-dimensional Abelian Higgs models

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is also interesting. The topological defects in two and three dimensions are instantons and particles (in four dimensions they are strings). The origin of the Aharonov–Bohm effect is the same as in the four-dimensional case: it is also due to the topological interaction between the particles and topological defects.

In Section 2 we repeat the calculations of Ref. [7] for four-dimensional Abelian Higgs model in the continuum limit and show explicitly the existence of the Aharonov–Bohm effect in the field theory. We rewrite the theory in terms of the string variables and study the interaction between the strings and particles of different charges.

Section 3 is devoted to the construction of the operator which creates the string in a given time slice on the contour \mathcal{C} . This operator is the continuum analogue of the lattice operator studied in [7, 10].

In Section 4 we consider the theory in the Hamiltonian formalism and show that the string creation operator has nontrivial commutation relations^a with the Wilson loop operator; this is a direct consequence of the Aharonov–Bohm effect. We give several other examples of nontrivial commutation relations.

In Section 5 for the lattice Abelian Higgs model for $D = 2, 3, 4$ we show the existence of the specific interaction of the test particles which is due to Aharonov–Bohm effect. Note, that the Aharonov–Bohm effect may be also important for the confinement problem in non-abelian gauge theories [11].

2. The Aharonov–Bohm Effect in the Abelian Higgs Model

We discuss the four-dimensional Abelian Higgs model with the Higgs bosons carrying the charge Ne , the partition function can be written down as follows:

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\Phi \exp \left\{ - \int d^4x \left[\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} |D_\mu \Phi|^2 + \lambda (|\Phi|^2 - \eta^2)^2 \right] \right\}, \quad (1)$$

where $D_\mu = \partial_\mu - iNeA_\mu$. The integration over the complex Higgs field $\Phi = |\Phi| \exp(i\theta)$ can be represented as the integration over the radial part of the Higgs field $|\Phi|$ and the integration over the phase θ . In this talk we consider the London limit, $\lambda \rightarrow \infty$, therefore the radial part of the field Φ is fixed and partition function (1) can be rewritten as:

$$\mathcal{Z} = \int \mathcal{D}A_\mu \mathcal{D}\theta \exp \left\{ - \int d^4x \left[\frac{1}{4} F_{\mu\nu}^2 + \frac{\eta^2}{2} (\partial_\mu \theta + NeA_\mu)^2 \right] \right\}, \quad (2)$$

where $\eta^2 = \langle |\Phi|^2 \rangle$. The functional integral over θ should be carefully treated, since θ is not defined on the manifolds where $Im\Phi = Re\Phi = 0$. These equations define the two-dimensional surfaces which are world sheets of the ANO strings; the Higgs field is zero at the center of the ANO string. The variable θ can be represented as the sum of the regular and the singular part, $\theta = \theta^r + \theta^s$. The singular part defines the ANO string degrees of freedom. The string world sheet current $\Sigma_{\mu\nu}$ and θ^s are related by the equations [12, 13]:

^aBy a nontrivial commutation relation we mean a relation of the type: $AB - e^{i\epsilon}BA = 0$.

$$\partial_{[\mu}, \partial_{\nu]} \theta^s(x, \tilde{x}) = 2\pi \epsilon_{\mu\nu\alpha\beta} \Sigma_{\alpha\beta}(x, \tilde{x}), \quad (3)$$

$$\Sigma_{\alpha\beta}(x, \tilde{x}) = \int_{\Sigma} d\sigma_{\alpha\beta}(\tilde{x}) \delta^{(4)}[x - \tilde{x}(\sigma)], \quad d\sigma_{\alpha\beta}(\tilde{x}) = \epsilon^{ab} \partial_a \tilde{x}_\alpha \partial_b \tilde{x}_\beta d^2\sigma,$$

where $\tilde{x} = \tilde{x}(\sigma)$ are the coordinates of the two-dimensional singularities parametrized by σ_a , $a = 1, 2$; $\partial_a = \frac{\partial}{\partial \sigma_a}$; $\theta^s(x, \tilde{x})$ is the function of x and the functional of \tilde{x} .

The integration over the fields θ^r and A in eq.(2) leads to the following string partition function [8]:

$$\mathcal{Z} = \int \mathcal{D}\tilde{x} J(\tilde{x}) \cdot \exp \left\{ -\eta^2 \pi^2 \int_{\Sigma} \int_{\Sigma} d\sigma_{\mu\nu}(\tilde{x}) \mathcal{D}_m^{(4)}(\tilde{x} - \tilde{x}') d\sigma_{\mu\nu}(\tilde{x}') \right\}, \quad (4)$$

where $\mathcal{D}_m^{(4)}(x)$ is the euclidean Yukawa propagator, $(\Delta + m^2)\mathcal{D}_m^{(4)}(x) = \delta^{(4)}(x)$, and $m^2 = N^2 e^2 \eta^2$ is the mass of the gauge boson. The action which enters the partition function (4) was already discussed in Ref.[12, 14]. The Jacobean $J(\tilde{x})$ is due to the change of the field variables to the string variables [8]. Due to presence of the Jacobean the conformal anomaly is absent and the quantum theory of strings (4) exists in four dimensions (see for details the discussion in Ref's. [8, 9]).

There exists a nontrivial long-range topological interaction of ANO strings with particles of charge Me provided the ration $\frac{M}{N}$ is non-integer. This interaction is the four-dimensional analogue [3, 4] of the Aharonov-Bohm effect studied for the lattice Abelian Higgs model in [7]. Here we show the existence of the topological terms in the string representation of the theory. Let us consider the Wilson loop for the particle of the charge Me :

$$W_M(\mathcal{C}) = \exp \left\{ iMe \int d^4x j_\mu^{\mathcal{C}}(x) A_\mu(x) \right\} = \exp \left\{ iMe \int_{\mathcal{C}} dx^\mu A_\mu(x) \right\}, \quad (5)$$

where the current is the δ -function on a contour \mathcal{C} :

$$j_\mu^{\mathcal{C}}(x) = \int_{\mathcal{C}} dt \dot{\tilde{z}}^\mu(t) \delta^{(4)}(x - \tilde{z}(t)) \quad (6)$$

and the function $\tilde{z}_\mu(t)$ parametrizes the contour.

Substituting (5) into the path integral (2) and changing the field variables to the string variables we obtain:

$$\begin{aligned} & \langle W_M(\mathcal{C}) \rangle = \\ & \frac{1}{\mathcal{Z}} \int \mathcal{D}\tilde{x} J(\tilde{x}) \exp \left\{ - \int d^4x \int d^4y \left[\pi^2 \eta^2 \Sigma_{\mu\nu}(x) \mathcal{D}_m^{(4)}(x - y) \Sigma_{\mu\nu}(y) \right. \right. \\ & \left. \left. + \frac{M^2 e^2}{2} j_\mu^{\mathcal{C}}(x) \mathcal{D}_m^{(4)}(x - y) j_\mu^{\mathcal{C}}(y) + \pi i \frac{M}{N} j_\mu^{\mathcal{C}}(x) \mathcal{D}_m^{(4)}(x - y) \partial_\nu \epsilon_{\mu\nu\alpha\beta} \Sigma_{\alpha\beta}(y) \right] \right. \\ & \left. + 2\pi i \frac{M}{N} \mathcal{L}(\Sigma, \mathcal{C}) \right\}, \quad (7) \end{aligned}$$

where $m = Ne\eta$ is the boson mass, and

$$\begin{aligned}\mathcal{L}(\Sigma, \mathcal{C}) &= \frac{1}{2} \int d^4x \int d^4y \epsilon_{\mu\nu\alpha\beta} \Sigma_{\mu\nu}(x) j_\alpha^{\mathcal{C}}(y) \partial_\beta \mathcal{D}_0^{(4)}(x-y) = \\ &= \frac{1}{4\pi^2} \int d^4x \int d^4y \epsilon_{\mu\nu\alpha\beta} \Sigma_{\mu\nu}(x) j_\alpha^{\mathcal{C}}(y) \frac{(x-y)_\beta}{|x-y|^4}\end{aligned}\quad (8)$$

is the linking number of the string world sheet Σ and the trajectory of the charged particle \mathcal{C} , this formula represents a four-dimensional analogue of the Gauss linking number for loops in three dimensions. The first three terms in the exponent in (7) are short range interactions and selfinteractions of strings and the tested particle. The forth term is the long-range interaction which describes the four-dimensional analogue [3, 4] of the Aharonov–Bohm effect: strings correspond to solenoids which scatter charged particles. \mathcal{L} is an integer, and if M/N is an integer too, then there is no long-range interaction; this situation corresponds to such a relation between the magnetic flux in the solenoid and the charge of the particle when the scattering of the charged particle is absent. The reason for this long-range interaction is that the charges $M = e, 2e, \dots (N-1)e$ cannot be completely screened by the condensate of the field of charge Ne ; if M/N is integer, then the screening is complete and there are no long-range forces. Note that the cloud of the screening charges screen the Coulomb interaction between test-particles, but do not affect the Aharonov–Bohm interaction [6].

Another consequence of the Aharonov–Bohm effect can be obtained, if we consider the operator $\mathcal{F}_N(\mathcal{S})$ [5] which creates the string with the magnetic flux $\frac{2\pi}{Ne}$ moving along a fixed closed surface \mathcal{S} . The operator $\mathcal{F}_N(\mathcal{S})$ is the analogue of the Wilson loop which creates the particle moving along the closed loop \mathcal{C} . An explicit form of $\mathcal{F}_N(\mathcal{S})$ is [5]:

$$\mathcal{F}_N(\mathcal{S}) = \exp \left\{ -\frac{\pi}{Ne} \int_{\mathcal{S}} d\sigma_{\mu\nu} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}(x) \right\}. \quad (9)$$

There exists an operator which can be calculated *exactly*, [4]; this operator is the normalized product of the Wilson loop $W_M(\mathcal{C})$ and $\mathcal{F}_N(\mathcal{S})$:

$$A_{NM}(\mathcal{S}, \mathcal{C}) = \frac{\mathcal{F}_N(\mathcal{S}) W_M(\mathcal{C})}{\langle \mathcal{F}_N(\mathcal{S}) \rangle \langle W_M(\mathcal{C}) \rangle}. \quad (10)$$

Here $\langle \mathcal{F}_N(\mathcal{S}) \rangle$ is a constant which depends on the regularization scheme. Substituting this operator into the functional integral (2) and integrating over the fields A and θ , we obtain the following result:

$$\langle A_{NM}(\mathcal{S}, \mathcal{C}) \rangle = e^{2\pi i \frac{M}{N} \mathcal{L}(\mathcal{S}, \mathcal{C})}. \quad (11)$$

The meaning of this result is very simple. If the surface \mathcal{S} lies in a given time slice, then $\langle A_{NM}(\mathcal{S}, \mathcal{C}) \rangle = \exp \left\{ \frac{2\pi i}{Ne} Q_{\mathcal{S}} \right\}$ (see [4, 5]), where $Q_{\mathcal{S}}$ is the total charge inside the volume bounded by the surface \mathcal{S} ; if $\mathcal{L}(\mathcal{S}, \mathcal{C}) = n$, then there is the charge Mne in the volume bounded by \mathcal{S} .

3. The String Creation Operator.

In the previous Section we have derived the partition function of the Abelian Higgs model as a sum over the closed world sheets of the ANO strings. Now we construct the operator which creates the string on a closed loop at a given time. The vacuum expectation value of this operator is the sum over all surfaces spanned on a given loop. A similar operator for the lattice theory was suggested in [10, 7]. The construction is similar to the soliton creation operator suggested by Fröhlich and Marchetti [15]. First we consider the model [13] which is dual to the original Abelian Higgs model. It contains the gauge field B_μ dual to the gauge field A_μ , and the hypergauge field $h_{\mu\nu}$ dual to θ^r . In order to get it we change the integration in θ^s to the integration over \tilde{x} and make the duality transformation, the details of this transformation are given in [13]. Taking into the account the Jacobian [8], we get:

$$\mathcal{Z} = \int \mathcal{D}h \mathcal{D}B \mathcal{D}\tilde{x} J(\tilde{x}) \exp \left\{ - \int d^4x \left[\frac{1}{3\eta^2} H_{\mu\nu\alpha}^2 + \frac{e^2 N^2}{2} (h_{\mu\nu} - \partial_\mu B_\nu + \partial_\nu B_\mu)^2 + 2\pi i h_{\mu\nu} \Sigma_{\mu\nu}(x, \tilde{x}) \right] \right\}, \quad (12)$$

where $H_{\mu\nu\sigma} = \partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} + \partial_\sigma h_{\mu\nu}$ is the field strength of the hypergauge field $h_{\mu\nu}$. The action of the dual theory is invariant under the gauge transformations: $B_\mu(x) \rightarrow B_\mu(x) + \partial_\mu \alpha(x)$, $h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x)$, and under the hypergauge transformations: $B_\mu(x) \rightarrow B_\mu(x) - \gamma_\mu(x)$, $h_{\mu\nu}(x) \rightarrow h_{\mu\nu}(x) + \partial_\mu \gamma_\nu(x) - \partial_\nu \gamma_\mu(x)$.

The ANO string carries magnetic flux, and in order to construct the creation operator, it is natural to use the dual Wilson loop: $\mathcal{W}_D(\mathcal{C}) = \exp\{i \int d^4x B_\mu(x) j_\mu^{\mathcal{C}}(x)\}$, where the current $j_\mu^{\mathcal{C}}(x)$ defines the loop \mathcal{C} (6). This operator is gauge invariant but it is not hypergauge invariant, and its vacuum expectation value is zero. To construct the hypergauge invariant operator [10, 7], we follow an idea of Dirac [16], who suggested the gauge invariant creation operator of a particle with the charge M :

$$\Phi_M^{\mathcal{C}}(\mathbf{x}) = \Phi_M(\mathbf{x}) \exp \left\{ i M e \int d^3y G_l(\mathbf{x} - \mathbf{y}) A_l(\mathbf{y}) \right\}, \quad (13)$$

here $\partial_i G_i(\mathbf{x}) = \delta^{(3)}(\mathbf{x})$, and the gauge variation of the matter field $\Phi(x) \rightarrow \Phi_M(x) e^{i M e \alpha(x)}$ is compensated by the gauge variation of cloud of photons A_μ . Now we use a similar construction, namely, we surround $\mathcal{W}_D(\mathcal{C})$ by the cloud of the Goldstone bosons:

$$U(\mathcal{C}) = \mathcal{W}_D(\mathcal{C}) \exp \left\{ \frac{i}{2} \int d^3y G_{\mathcal{C}}^{ij}(x - y) h_{ij}(y) \right\}. \quad (14)$$

It is easy to see that $U(\mathcal{C})$ is hypergauge invariant if the skew-symmetrical tensor $G_{\mathcal{C}}^{ij}(x)$ satisfies the equation $\partial_i G_{\mathcal{C}}^{ik}(x) = j_k^{\mathcal{C}}(x)$. It is convenient to choose $G_{\mathcal{C}}^{ik}(x)$ as the surface,

^bIn this and in the next sections, the Latin indices vary from 1 to 3 and the Greek indices vary from 0 to 3.

spanned on the loop \mathcal{C} : $G_{\mathcal{C}}^{ij} = \int_{S_{\mathcal{C}}} d\sigma^{ij}(\tilde{x})\delta^{(4)}[x - \tilde{x}(\sigma)]$ (cf. eq.(3)). Since the string creation operator should act at a definite time slice, the surface defined by $G_{\mathcal{C}}^{ij}(x)$ and the loop \mathcal{C} should belong to that time slice^c.

Substituting the operator (14) into the dual partition function (12) and performing the inverse duality transformation, we get the vacuum expectation value of the string creation operator in terms of the original fields A and θ :

$$\begin{aligned} \langle U(\mathcal{C}) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}A \mathcal{D}\theta \exp \left\{ - \int d^4x \left[\frac{1}{4} \left(F_{\mu\nu} + \frac{2\pi}{Ne} \epsilon_{\mu\nu\lambda\sigma} G_{\mathcal{C}}^{\lambda\sigma}(x) \right)^2 \right. \right. \\ \left. \left. + \frac{\eta^2}{2} (\partial_\mu \theta + Ne A_\mu)^2 \right] \right\}, \end{aligned} \quad (15)$$

where the tensor $G_{\mathcal{C}}^{\mu\nu}$ is equal to $G_{\mathcal{C}}^{ij}$ if $\mu = i$ and $\nu = j$ are spatial indices, and $G_{\mathcal{C}}^{0\mu} = G_{\mathcal{C}}^{\mu 0} = 0$ for any μ . If we change the field variables in (15) to the string variables, we get a sum over the closed surfaces Σ :

$$\begin{aligned} \langle U(\mathcal{C}) \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\tilde{x} J(\tilde{x}) \exp \left\{ -\eta^2 \pi^2 \int d^4x \int d^4y [(\Sigma^{\mu\nu}(x, \tilde{x}) + \right. \\ \left. G_{\mathcal{C}}^{\mu\nu}(x)) \mathcal{D}_m^{(4)}(x - y) (\Sigma^{\mu\nu}(y, \tilde{x}) + G_{\mathcal{C}}^{\mu\nu}(y))] \right\}, \end{aligned} \quad (16)$$

where $\Sigma^{\mu\nu}(x, \tilde{x})$ is defined by eq.(3).

The summation over all closed surfaces $\Sigma^{\mu\nu}$, plus the open surface $G^{\mu\nu}$ with the boundary \mathcal{C} , is equivalent to the summation over all closed surfaces and over *all* surfaces spanned on the loop \mathcal{C} . Therefore, the operator $U(\mathcal{C})$ creates a string on the loop \mathcal{C} . Using the string creation operators, it is easy to construct the operators which correspond to the processes of decay and scattering of the strings.

4. Aharonov–Bohm Effect In The Hamiltonian Formalism.

In this section we consider the ANO strings in the framework of the canonical quantization. We start with the standard commutation relations: $[\pi^i(\mathbf{x}), A^j(\mathbf{y})] = -i\delta_{ij}\delta(\mathbf{x} - \mathbf{y})$, $\pi^i = F^{0i}$ and $[\pi_\phi(\mathbf{x}), \phi(\mathbf{y})] = -i\delta(\mathbf{x} - \mathbf{y})$, $\pi_\phi = (D^0\phi)^*$. Using the string creation operators (9) and (14), we construct several operators, which satisfy the commutator relations of the type: $A \cdot B - B \cdot Ae^{i\xi} = 0$. Similar operators are known for 3D Abelian models, see for example refs.[17]. The physical phenomenon leading to the nontrivial commutation relations in the nonabelian theories was discussed by 't Hooft [18].

First, let us consider the operator $U_{str}(\mathcal{C})$ which creates the ANO string on the loop \mathcal{C} :

$$U_{str}(\mathcal{C}) = \exp \left\{ \frac{2\pi i}{Ne} \int d^3x \frac{1}{2} \epsilon_{ijk} G_{\mathcal{C}}^{ij}(x) \pi^k(x) \right\}, \quad (17)$$

^cThe solution of the equation $\partial_i G_{\mathcal{C}}^{ik}(x) = j_k^{\mathcal{C}}(x)$ is non-unique, moreover we choose a two dimensional surface as the support of G^{ik} , the solution which has three-dimensional support can be of the form: $G_{\mathcal{C}}^{ik} = \int d^3y \partial^{[i} j_{\mathcal{C}}^{k]}(y) \mathcal{D}_0^{(3)}(\mathbf{x} - \mathbf{y})$, where $\mathcal{D}_0^{(3)} = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|}$. It is easy to find that all these ambiguities do not change the physical results.

here $G_{\mathcal{C}}^{ij}(x)$ is the same function as in eq.(14). The operator (17) is a special case of the creation operator:

$$U[A^{cl}] = \exp \left\{ i \int d^3x A_k^{cl}(x) \pi^k(x) \right\}, \quad (18)$$

where $A^{cl}(x)$ is a classical field. It is easy to see, that $U[A^{cl}]|A(x)\rangle = |A(x) + A^{cl}(x)\rangle$. In (17) we have $A_k^{cl}(x) = \frac{2\pi}{Ne} \epsilon_{ijk} G_{\mathcal{C}}^{ij}(x)$, and the magnetic field corresponds to the infinitely thin string on the loop \mathcal{C} : $B_i(x) = \frac{2\pi}{Ne} j_i^{\mathcal{C}}(x)$; the current $j_i^{\mathcal{C}}$ is defined by eq. (6).

The commutation relations for the operator (17) with the operators of the electric charge $Q = \int d^3x \partial_i \pi^i(\mathbf{x})$ and the magnetic flux $\Phi_i = \int d^3x \epsilon_{ijk} \partial^j A^k(\mathbf{x})$ also show that $U_{str}(\mathcal{C})$ creates a string which carries the magnetic flux $\frac{2\pi}{Ne}$ on the contour \mathcal{C} :

$$[Q(x_0, \mathbf{x}), U_{str}(\mathcal{C})] = 0, \quad [\Phi^j(x_0, \mathbf{x}), U_{str}(\mathcal{C})] = \frac{2\pi}{Ne} j_i^{\mathcal{C}}(x) U_{str}(\mathcal{C}). \quad (19)$$

Note that, the string creation operator (15) considered in the previous section can be rewritten in the following way:

$$U(\mathcal{C}) = \exp \left\{ - \int d^4x \left[\frac{1}{4} \left(F_{\mu\nu} + \frac{2\pi}{Ne} \epsilon_{\mu\nu\lambda\sigma} G_{\mathcal{C}}^{\lambda\sigma}(x) \right)^2 - \frac{1}{4} F_{\mu\nu}^2 \right] \right\}, \quad (20)$$

and it is clear that, up to an inessential factor, it coincides with (17).

Now we consider the commutator of the operator $U_{str}(\mathcal{C}_1)$ and the Wilson loop $W_M(\mathcal{C}_2)$ (5), the contours \mathcal{C}_1 and \mathcal{C}_2 belong to the same time slice. Using the relation $e^A e^B = e^B e^A e^{[A,B]}$, which is valid if $[A, B]$ is a c -number, it is easy to get:

$$U_{str}(\mathcal{C}_1) W_M(\mathcal{C}_2) - e^{i\xi(\mathcal{C}_1, \mathcal{C}_2)} W_M(\mathcal{C}_2) U_{str}(\mathcal{C}_1) = 0, \quad (21)$$

where $\xi(\mathcal{C}_1, \mathcal{C}_2) = \frac{2\pi M}{N} \mathcal{L}(\mathcal{C}_1, \mathcal{C}_2)$, and $\mathcal{L}(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{4\pi} \int_{\mathcal{C}_1} dx_i \int_{\mathcal{C}_2} dy_j \epsilon^{ijk} \frac{(x-y)_k}{|\mathbf{x}-\mathbf{y}|^3}$ is the linking number of the loops \mathcal{C}_1 and \mathcal{C}_2 , see Fig.1. The commutation relation (21) is the direct consequence of the Aharonov–Bohm effect; the wave function of the particle of the charge Me acquires the additional phase $e^{\frac{2\pi i M}{N}}$ if it goes around a solenoid with the magnetic flux $\frac{2\pi}{Ne}$.

The next example is the commutation relation of the Dirac operator $\Phi_M^c(\mathbf{x})$ (13) which creates the particle with charge M at the point \mathbf{x} , and the operator $\mathcal{F}_N(\mathcal{S})$ which creates the string on the surface \mathcal{S} . In Minkowsky space, the operator $\mathcal{F}_N(\mathcal{S})$ has the form (an analogue of eq.(9)):

$$\mathcal{F}_N(\mathcal{S}) = \exp \left\{ \frac{i\pi}{Ne} \int_{\mathcal{S}} d\sigma_{\mu\nu} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}(x) \right\}. \quad (22)$$

If the surface \mathcal{S} belongs to the same time slice as the point \mathbf{x} , then:

$$\mathcal{F}_N(\mathcal{S}) \Phi_M^c(\mathbf{x}) - \Phi_M^c(\mathbf{x}) \mathcal{F}_N(\mathcal{S}) e^{i\theta(\mathcal{S}, \mathbf{x})} = 0, \quad (23)$$

where $\theta(\mathcal{S}, \mathbf{x}) = \frac{2\pi M}{N} \Theta(\mathcal{S}, \mathbf{x})$. The function $\Theta(\mathcal{S}, \mathbf{x})$ is the "linking number" of the surface \mathcal{S} and the point \mathbf{x} , see Fig.2:

$$\Theta(\mathcal{S}, \mathbf{x}) = \begin{cases} 1 & \text{if } x \text{ lies inside volume bounded by } \mathcal{S}; \\ 0 & \text{otherwise} \end{cases}. \quad (24)$$

It is obvious that the commutation relation (23) is also a consequence of the Aharonov–Bohm effect.

Consider now the composite operator

$$H_{MN}(\mathbf{x}, \mathcal{S}) = \Phi_M^c(\mathbf{x}) \mathcal{F}_N(\mathcal{S}), \quad (25)$$

where the surface \mathcal{S} lies at the same time slice as the point \mathbf{x} . Using commutation relation (23) it is easy to find that:

$$H_{M_1, N}(\mathbf{x}_1, \mathcal{S}_1) H_{M_2, N}(\mathbf{x}_2, \mathcal{S}_2) - H_{M_2, N}(\mathbf{x}_2, \mathcal{S}_2) H_{M_1, N}(\mathbf{x}_1, \mathcal{S}_1) e^{i\zeta_{12}} = 0, \quad (26)$$

where $\zeta_{12} = \frac{2\pi M_1}{N} \Theta(\mathbf{x}_1, \mathcal{S}_2) - \frac{2\pi M_2}{N} \Theta(\mathbf{x}_2, \mathcal{S}_1)$. If the point \mathbf{x}_1 lies in the volume bounded by \mathcal{S}_2 , the point \mathbf{x}_2 lies out of the volume \mathcal{S}_1 , $M_{1,2} = 1$ and $N = 2$, then eq.(26) leads to the fermion-like commutation relation

$$H(\mathbf{x}_1)H(\mathbf{x}_2) + H(\mathbf{x}_2)H(\mathbf{x}_1) = 0, \quad (27)$$

where $H(\mathbf{x}_i) = H_{M_i}(\mathbf{x}_i, \mathcal{S}_i)$.

The commutation relations (26) and (27) can be explained as follows. The operator $\mathcal{F}_N(\mathcal{S})$ creates the closed world sheet of the ANO string and the configuration space of the (charged) particles becomes not simply connected. Similar reasons lead to nontrivial statistics in $2 + 1$ dimensions [19]. Note that all operators and commutation relations considered in the present section can be constructed in the free theory, but the states created by the operators $U_{str}(\mathcal{C}_1)$ and $\mathcal{F}_N(\mathcal{S})$ are unstable in this case. In the Abelian Higgs theory, the ANO strings exist as a solution of the classical equations of motion, and this fact justifies the study of the commutation relations which contain string creation operators.

5. Aharonov–Bohm Effect on the Lattice

Now we study the Aharonov–Bohm effect on the lattice. For the sake of convenience we use the differential forms formalism on the lattice (see Ref.[20] for the introduction). As in the previous Sections we consider the Abelian Higgs model in the London limit. The corresponding partition function in the Villain form is given by the following formula:

$$\mathcal{Z} = \int_{-\infty}^{+\infty} \mathcal{D}A \int_{-\pi}^{+\pi} \mathcal{D}\varphi \sum_{l(c_1) \in \mathbb{Z}} \exp \left\{ -\beta \|dA\|^2 - \gamma \|d\varphi + 2\pi l - NA\|^2 \right\}, \quad (28)$$

where A is the noncompact gauge field, ϕ is the phase of the Higgs field, l is the integer-valued one-form and we do not specify the dimension D of the space-time. In order to rewrite the partition function (28) in terms of topological defects we should perform the so

called Berezinski–Kosterlitz–Thaules (BKT) transformation [21] with respect to the form l . First change the summation variable, $\sum_{l(c_1) \in \mathbb{Z}} = \sum_{\substack{\sigma(c_2) \in \mathbb{Z} \\ d\sigma=0}} \sum_{q(c_0) \in \mathbb{Z}}$, here $l = m[\sigma] + dq$ and

$m[\sigma]$ is a particular solution of the equation $dm[\sigma] = \sigma$. Using the Hodge decomposition $m[\sigma] = \delta\Delta^{-1}\sigma + d\Delta^{-1}\delta m[\sigma]$ we introduce the noncompact field $\varphi_{n.c.} = \varphi + 2\pi(\Delta^{-1}\delta m[\sigma] + q)$, $\sum_{q(c_0) \in \mathbb{Z}} \int_{-\pi}^{+\pi} \mathcal{D}\varphi = \int_{-\infty}^{+\infty} \mathcal{D}\varphi_{n.c.}$, the integration over the noncompact fields A and $\varphi_{n.c.}$ gives:

$$\mathcal{Z}^{BKT} = \text{const.} \sum_{\substack{* \sigma(*c_{D-2}) \in \mathbb{Z} \\ \delta^* \sigma = 0}} \exp \left\{ -4\pi^2 \gamma \left(\sigma, (\Delta + m^2)^{-1} \sigma \right) \right\}, \quad (29)$$

where $m^2 = N^2 \gamma / \beta$ is the mass of the vector boson field. The closed $(D-2)$ -forms σ represents the world paths for the topological defects of the theory, which are instanton-like defects in $D=2$, particle-like defects in $D=3$ and ANO strings in $D=4$ (compare the lattice equation (29) for $D=4$ with eq.(4) in the continuum).

The same transformations applied to the quantum average of the Wilson loop $W_M(\mathcal{C}) = \exp\{iM(A, j_{\mathcal{C}})\}$ leads to the following formula:

$$\begin{aligned} \langle W_M(\mathcal{C}) \rangle = & \frac{1}{\mathcal{Z}^{BKT}} \sum_{\substack{* \sigma(*c_{D-2}) \in \mathbb{Z} \\ \delta^* \sigma = 0}} \exp \left\{ -4\pi^2 \gamma \left({}^* \sigma, (\Delta + m^2)^{-1} {}^* \sigma \right) \right. \\ & \left. - \frac{M^2}{4\gamma} (j_{\mathcal{C}}, (\Delta + m^2)^{-1} j_{\mathcal{C}}) - 2\pi i \frac{M}{N} \left(j_{\mathcal{C}}, (\Delta + m^2)^{-1} \delta \sigma \right) + 2\pi i \frac{M}{N} \mathcal{L}(\sigma, j_{\mathcal{C}}) \right\}, \quad (30) \end{aligned}$$

The first three terms in this expression are short-range Yukawa forces between defects. The last term is long-range and it has the topological origin: $\mathcal{L}(\sigma, j_{\mathcal{C}})$ is the linking number between the world trajectories of the defects σ and the Wilson loop $j_{\mathcal{C}}$:

$$\mathcal{L}(\sigma, j_{\mathcal{C}}) = ({}^* j_{\mathcal{C}}, \Delta^{-1} d^* \sigma). \quad (31)$$

\mathcal{L} is equal to the number of points at which the loop $j_{\mathcal{C}}$ intersects the $(D-1)$ -dimensional volume *n bounded by the closed $(D-2)$ -manifold defined by the form ${}^* \sigma(*c_{D-2})$, see [7] for $D=4$. For four dimensions eq.(31) is the lattice analogue of eq.(8). Note that eq.(30) for the quantum average of the Wilson loop remains unchanged under the transformation^d $N \rightarrow 1$, $M \rightarrow \frac{M}{N}$, $\beta \rightarrow \frac{\beta}{N^2}$.

Let us first consider the quantum average (30) in two dimensions. In the limit $\gamma \gg \beta$ (the vector boson mass in the lattice units is much greater than unity) this equation reduces to

$$\langle W_M(\mathcal{C}) \rangle = \frac{1}{\mathcal{Z}^{BKT}} \sum_{\substack{* k(*c_0) \in \mathbb{Z} \\ \delta^* k = 0}} \exp \left\{ -\frac{4\pi^2 \beta}{N^2} \|{}^* k\|^2 + 2\pi i \frac{M}{N} \mathcal{L}(k, j_{\mathcal{C}}) \right\}, \quad (32)$$

^dThis is valid only for the Abelian Higgs Model with the noncompact gauge field A .

where the zero-form $*k$ represents the instanton "world paths" and we omitted the terms of the order $O(m^{-2})$.

The linking number can be rewritten as $\mathcal{L}(k, j_{\mathcal{C}}) = (k, m_{\mathcal{C}})$, where two-form $m_{\mathcal{C}}$ represents the surface spanned on the contour \mathcal{C} , and the result is:

$$\langle W_M(\mathcal{C}) \rangle = \left(\frac{\sum_{j \in \mathbb{Z}} \exp \left\{ -\frac{4\pi^2\beta}{N^2} j^2 + 2\pi i \frac{M}{N} j \right\}}{\sum_{j \in \mathbb{Z}} \exp \left\{ -\frac{4\pi^2\beta}{N^2} j^2 \right\}} \right)^{\mathcal{S}(\mathcal{C})} = \text{const.} \exp \{ -\sigma(M, N; \gamma) \cdot \mathcal{S}(\mathcal{C}) \}, \quad (33)$$

where $\mathcal{S}(\mathcal{C})$ is the area of the surface which is spanned on the contour \mathcal{C} .

The string tension $\sigma(M, N; \gamma)$ is given by the equation:

$$\sigma(M, N; \gamma) = -\ln \left[\frac{\Theta \left(\frac{4\pi^2\beta}{N^2}, \frac{M}{N} \right)}{\Theta \left(\frac{4\pi^2\beta}{N^2}, 0 \right)} \right], \quad (34)$$

where $\Theta(x, q)$ is Θ -function:

$$\Theta(x, q) = \sum_{j \in \mathbb{Z}} \exp \left\{ -x j^2 + 2\pi i q j \right\}. \quad (35)$$

Note, that the string tension σ has interesting property: if the charge of the test particle is completely screened by the charge of the Higgs condensate $\left(\frac{M}{N} \in \mathbb{Z} \right)$, the string tension is zero, since $\Theta(x, q + 2\pi n) = \Theta(x, q)$ where n is integer. The dependence of the string tension $\sigma(M, N; \gamma)$ on β for $M = 1$ and $N = 2, 3, 4$ is shown on Fig.3. The area law of the Wilson loops for fractionally charged particles in the $2D$ abelian Higgs model was first found in Ref. [22], but it was not realized that the nonzero value of the string tension is due to the analogue of the Aharonov–Bohm effect.

Now we consider the three-dimensional Abelian Higgs model in the limit $\gamma \gg \beta$. The analogue of eq.(33) is given by the formula:

$$\langle W_M(\mathcal{C}) \rangle = \frac{1}{\mathcal{Z}^{BKT}} \sum_{\substack{*j(*c_1) \in \mathbb{Z} \\ \delta^* j = 0}} \exp \left\{ \frac{4\pi^2\beta}{N^2} \|*j\|^2 + 2\pi i \frac{M}{N} (*m_{\mathcal{C}}, *j) \right\}, \quad (36)$$

where the one-form $*j$ represents the world trajectories of the vortices and $m_{\mathcal{C}}$ is again an arbitrary surface, which is spanned on the contour \mathcal{C} .

Let us represent the condition of closeness of the vortex lines j in eq.(36) by the introduction of the integration over additional compact form C :

$$\sum_{\substack{*j(*c_1) \in \mathbb{Z} \\ \delta^* j = 0}} \dots = \sum_{*j(*c_1) \in \mathbb{Z}} \delta(\delta^* j) \dots = \int_{-\pi}^{+\pi} \mathcal{D}^* C \sum_{*j(*c_1) \in \mathbb{Z}} \exp \{ i(\delta^* j, *C) \} \dots \quad (37)$$

Inserting the unity

$$1 = \int_{-\infty}^{+\infty} \mathcal{D}^* F \delta(*F - *j) \quad (38)$$

in eq.(36), we get:

$$\begin{aligned} \langle W_M(\mathcal{C}) \rangle = & \frac{1}{\mathcal{Z}^{BKT}} \int_{-\infty}^{+\infty} \mathcal{D}^* C \int_{-\infty}^{+\infty} \mathcal{D}^* F \sum_{*j(*c_1) \in \mathbb{Z}} \cdot \\ & \exp \left\{ -\frac{4\pi^2\beta}{N^2} \|*F\|^2 + 2\pi i \frac{M}{N} (*m_C, *F) + i(\delta^* F, *C) \right\} \cdot \delta(*F - *j) . \end{aligned} \quad (39)$$

The use of the Poisson formula $2\pi \sum_{n \in \mathbb{Z}} \delta(n - x) = \sum_{n \in \mathbb{Z}} e^{inx}$ and integration out of the fields $*F$ leads to the dual representation of the quantum average (36):

$$\langle W_M(\mathcal{C}) \rangle = \frac{1}{\mathcal{Z}^{BKT}} \sum_{*j(*c_1) \in \mathbb{Z}} \int_{-\infty}^{+\infty} \mathcal{D} C \exp \left\{ -\frac{N^2}{16\pi^2\beta} \|d^* C + 2\pi \frac{M}{N} *m_C + 2\pi *j\|^2 \right\} . \quad (40)$$

At $\beta \rightarrow 0$ we can evaluate this expression in the semiclassical approximation, the result is:

$$\langle W_M(\mathcal{C}) \rangle = \text{const.} \exp \left\{ -\frac{q^2 N^2}{4\beta} (j_C, \Delta^{-1} j_C) \right\} , \quad (41)$$

where

$$q = \min_{K \in \mathbb{Z}} \left| \frac{M}{N} - K \right| . \quad (42)$$

Let us consider the large Wilson loop of the size $T \times R$, $T \gg R \gg 1$. Taking into account that two-dimensional massless propagator $\Delta^{-1}(x)$ behaves at large x as $\Delta^{-1}(x) = C_0 \cdot \ln x + \dots$, where C_0 is the numerical constant, we find in the leading order:

$$\langle W_M(\mathcal{C}) \rangle = \text{const.} \exp \{ -\kappa(M, N; \beta) \cdot T \ln R \} , \quad (43)$$

where the coefficient $\kappa(M, N; \beta)$ is given by the formula:

$$\kappa(M, N; \beta) = C_0 \frac{q^2 N^2}{4\beta} , \quad (44)$$

q is defined by eq.(42).

We see, that the Aharonov–Bohm effect leads to the logarithmic potential in 3D for the test particles of the charge Me . If $\frac{M}{N}$ is integer (complete screening) this potential vanishes. The logarithmic behavior of the potential was also predicted from some simple considerations in Ref. [23].

In the four-dimensional theory the Aharonov–Bohm interaction leads to corrections to the perimeter law (*i.e.* to mass renormalization). Nevertheless this effect can be important at finite temperature. When the temperature increases the system becomes closer to $3D$ theory and at some temperatures the ANO strings give rise to the logarithmic term in the potential extracted from the spatial Wilson loop. The behavior of the Wilson loops for the lattice $4D$ Abelian Higgs model with the compact gauge field at finite temperature was studied both analytically [24] and numerically [25]. In the phase diagram of compact version of the theory the confining region exists and therefore the logarithmic behavior is difficult to investigate. We are planning to investigate numerically the noncompact Abelian Higgs model at finite temperature in which the logarithmic potential should not be suppressed.

In this section we found several formulae, eqs.(34,35) and eqs.(42,44), which depend on the fractional part of the ratio $\frac{M}{N}$. This remarkable dependence is due to the Aharonov–Bohm effect.

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Figure Captions

Fig.1. Linking of the contours \mathcal{C}_1 and \mathcal{C}_2 in $3D$;

Fig.2. Linking of the surface \mathcal{S} and the point \mathbf{x} in $3D$;

Fig.3. The dependence of the string tension $\sigma(M, N; \gamma)$ (eq.(34)) on β for $M = 1$ and $N = 2, 3, 4$

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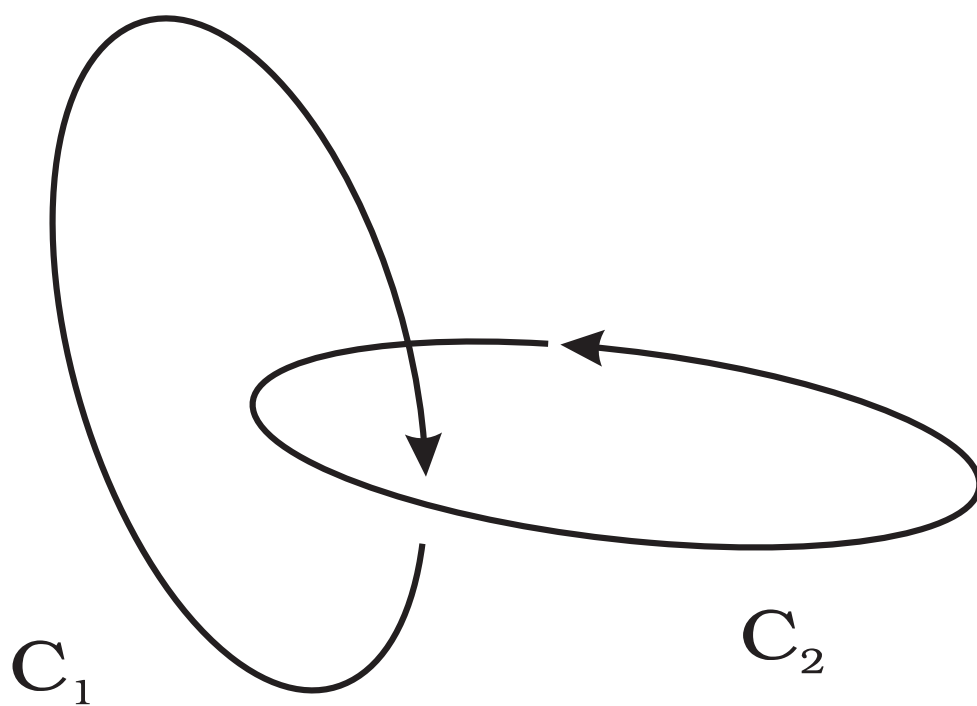


Fig. 1

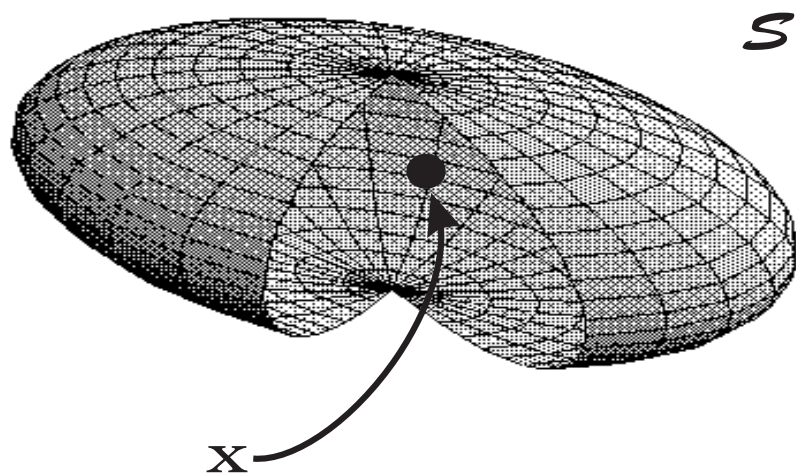


Fig.2

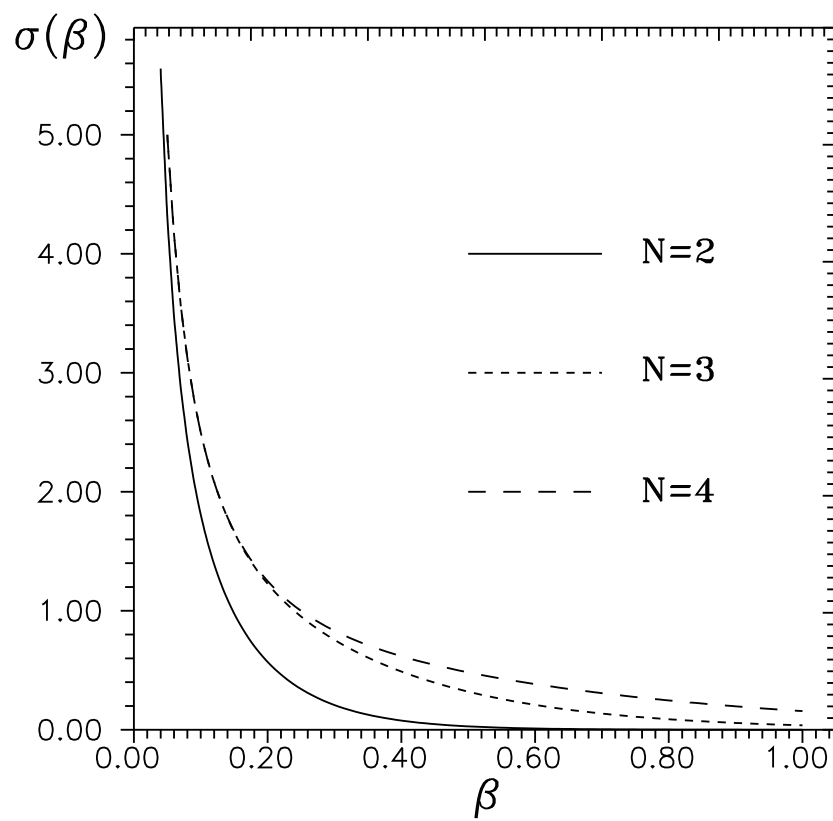


Fig.3